Approximate Frequency Domain Analysis for Linear Damped Space Structures

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A method is presented for the analysis of damped structures in which the structural components are represented by impedance models and analyzed in the frequency domain. Methods are presented to assemble and condense system impedance matrices and then to identify approximate mass, stiffness, and viscous damping matrices for systems whose impedances are complicated functions of frequency. Formulas are derived for determination of approximate values for the natural frequencies and damping of systems represented by mass, stiffness, and viscous damping matrices. The sensitivities of these approximate values to system parameter changes are analyzed. The implementation of these analysis tools is discussed and applied to a simple mechanical system.

Nomenclature

C	= damping matrix
E	= Young's modulus
F	= vector of global forces
K	= stiffness matrix
M	= mass matrix
P	= real power
T	= transformation matrix for reduction of coordinates
v	= vector of global velocities
δv	= first variation of the velocity mode shape
$W(s)$	= complex weighting function
X	= reactive power
Z	= impedance matrix
α_m, β_n	= viscoelastic material parameters
ζ	= damping ratio
η	= loss factor
ω_n	= natural frequency

Subscripts

E	= eliminated degrees of freedom (DOF)
EXT	= external
I	= imaginary
L	= linear
NL	= nonlinear
R	= real or retained DOF
V	= viscoelastic

Introduction

THE ability to effectively model passive damping, in a framework suitable for control design and system trades is a necessary technology in the design of controlled structures.[1] Compared with other aspects of dynamic modeling, there has been less formalism in the analysis and design of damping enhancement for structures. Passive damping design has for the most part been largely empirical with assessment resting heavily on structural testing. Since accurate 1-G ground vibration testing is difficult for space structures,[2] it would be highly desirable to develop an accurate and unified design methodology using component information.

One of the challenges in developing a unified methodology is that there are a number of sources of passive damping in space structures. The most prevalent is material damping. Damping is also inherent in the friction and impacting, which occur in the structural joints. The inherent damping in a space structure can be increased by using damping enhancement schemes.[3] These typically involve the addition of viscoelastic damping treatments,[4] proof mass dampers,[5] friction dampers,[6] and shunted piezoelectric materials[7] to the structure. Difficulties arise in determining the relative effectiveness of these techniques in system trade studies.

A logical step in the development of a passive damping methodology is consolidation of the sundry and disparate damping enhancement mechanisms into a common analytical framework. In this work such a framework is developed based on the frequency domain representation of systems in terms of their impedance matrices. The impedance matrix of the system can be assembled from its component impedances and analyzed for quantities such as kinetic energy, strain energy, and energy dissipation.

The frequency domain techniques for analysis of damped mechanical systems presented in this paper offer distinct advantages over conventional time domain analyses of damped systems. First, they are general enough to incorporate strain energy damping elements which have a more complicated frequency dependence than simple viscous or hysteretic damping. The problems associated with time domain modeling of viscoelastics whose inherent frequency dependence is represented by fractional derivatives[8] are overcome. Second, unlike modal strain energy analyses,[9] the frequency domain approach allows the consistent modeling of strain and kinetic dampers as well as continuous[10] or resonant structural components. Additionally, the framework allows a parallelism between analytical modeling and experimentally determined structural impedances of components. A complicated component can be characterized experimentally by its impedance matrix, which can be incorporated exactly into the system model without the need for an intermediate state space representation. In this way, a combination of analytical models and experimentally determined component information can be used in the analysis. Finally, approximate formulas for damping ratio and loss factor facilitate preliminary damping design without the solution of a generalized eigenvalue problem.

In the following sections, a frequency domain expression of force equilibrium is coupled with the traditional concept of structural impedance to model damped structures. This impedance model is used to develop a two-step method of approximate analysis. The first step is to derive approximate system mass, stiffness, and damping matrices from the impedance matrix. The second step is to use these matrices to derive approximate expressions for system resonant frequen-

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cies and damping, which are similar to Rayleigh's quotients. The sensitivity of system dynamic characteristics to variations in key system parameters will be evaluated. These tools can be used for the analysis and design of complex damped systems. A simple example problem will be analyzed to demonstrate the procedure.

Modeling Structures in the Frequency Domain

Impedance Modeling of Structural Components

In this section those issues relating to modeling structures in the frequency domain will be addressed. It will be assumed that a structure can be represented by forces and displacements or velocities at a set of global degrees of freedom (DOF) of the structure. At each of these DOF, force equilibrium is maintained between the applied external forces and the internal structural forces due to linear and nonlinear sources:

$$f_L(t) + f_{NL}(t) = f_{\text{EXT}}(t) \tag{1}$$

Taking the Laplace transform of Eq. (1)

$$\boldsymbol{F}_L(s) + \boldsymbol{F}_{NL}(s) = \boldsymbol{F}_{\text{EXT}}(s) \tag{2}$$

It is convenient to introduce the concept of mechanical impedance to represent the linear structural forces in Eq. (2). Mechanical impedance models the constitutive relations of a linear component by relating the force and velocity in that component. The impedance can be relatively simple, as for discrete springs or dampers, or a complicated function of frequency as is the case with viscoelastics or resonant subsystems. In general, the linear forces can be expressed as

$$\boldsymbol{F}_L(s) = \boldsymbol{Z}_L(s) \cdot \boldsymbol{v}(s) \tag{3}$$

where $\boldsymbol{F}_L$ is the vector of linear structural forces, $\boldsymbol{v}$ is the vector of global velocities, and $\boldsymbol{Z}_L$ is the global impedance matrix. Ignoring nonlinear structural forces, Eq. (2) can be written as

$$\boldsymbol{Z}_L(s) \cdot \boldsymbol{v}(s) = \boldsymbol{F}_{\text{EXT}}(s) \tag{4}$$

The global impedance matrix is assembled from the impedance matrices of the structural components in a manner directly analogous to the assembly of the stiffness matrices in the finite element method.[11] For an ordinary structural component with mass, viscous damping, and stiffness terms, the impedance can be represented as

$$\boldsymbol{Z}(s) = \boldsymbol{M}s + \boldsymbol{C} + \frac{\boldsymbol{K}}{s} \tag{5}$$

Thus the structural impedance of an ordinary component is a function of the complex Laplace parameter s and contains real and imaginary parts.

The impedance of structural components such as viscoelastic materials or substructures with component resonances can have much more complicated frequency dependence. As a case in point, the fractional calculus model of a viscoelastic material[8] allows a frequency dependent complex modulus of the form

$$E_V(s) = \frac{E_0\left(1 + \sum_n \beta_n s^{b_n}\right)}{\left(1 + \sum_m \alpha_m s^{a_m}\right)} \tag{6}$$

where constants α_m and β_n are typically obtained by fitting experimental data, and the exponents a_m and b_n are not necessarily integers. These nonintegral exponents lead to great difficulties in the time domain modeling of such materials since the fractional powers of s imply fractional derivatives in time. In the frequency domain, the impedance is represented as an algebraic function of frequency, which is easily incorporated into the model.

Frequency domain modeling also allows the inclusion of substructures, which include component resonances, without the necessity of modeling the component dynamics. A resonant substructure can be represented by an impedance matrix, which reflects the component dynamics as observed at a set of predefined interface points. Such substructure matrices can be derived from the reduction of other analytical models by inclusion of raw experimental data in the form of lookup tables or by fitting experimental transfer function data in the frequency domain with poles and zeros.

Reduction of Systems Represented by Impedance Matrices

Assembling the global impedance matrix of a structure from the impedances of the substructures leads to the model of Eq. (3), which can be solved at a given frequency for the total system response to a prescribed external forcing. If external forcing is not present at all DOF, the order of the system matrix can be reduced by eliminating the unforced DOF using matrix condensation.[12] Alternately, linearly dependent DOF can be reduced using a similarity transform between the dependent and independent DOF.

For the first reduction method, matrix condensation is performed upon the global impedance matrix by assuming that there is a set of DOF at which no forces act. The system can be partitioned

$$\begin{bmatrix} \boldsymbol{F}_R \\ 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{Z}_{RR} & \boldsymbol{Z}_{RE} \\ \boldsymbol{Z}_{ER} & \boldsymbol{Z}_{EE} \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_R \\ \boldsymbol{v}_E \end{bmatrix} \tag{7}$$

Following the technique of Ref. 12, a reduced impedance matrix can be defined:

$$\boldsymbol{F}_R = \boldsymbol{Z}'_L \boldsymbol{v}_R = [\boldsymbol{Z}_{RR} - \boldsymbol{Z}_{RE}\boldsymbol{Z}_{EE}^{-1}\boldsymbol{Z}_{ER}]\boldsymbol{v}_R \tag{8}$$

Since no information is discarded, this method is an exact reduction of the global impedance matrix.

The second method of impedance matrix reduction is to assume linear dependence among a subset of the global DOF of the form

$$\begin{bmatrix} \boldsymbol{v}_R \\ \boldsymbol{v}_E \end{bmatrix}_{n\times 1} = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{T}' \end{bmatrix}_{n\times m} \cdot \boldsymbol{v}_R = \boldsymbol{T} \cdot \boldsymbol{v}_R \tag{9}$$

Using the relationship to perform a change of variables, the reduced impedance matrix can be calculated as

$$\boldsymbol{Z}'_L = \boldsymbol{T}^H \boldsymbol{Z}_L \boldsymbol{T} \tag{10}$$

where $\boldsymbol{T}$ is a complex transformation matrix, and the superscript $(\ )^H$ represents the hermetian. These two techniques can be useful in reducing large global impedance matrices using the knowledge that there are no forces at certain DOF or that certain DOF are linearly dependent on a subset of the global DOF.

Approximate *M*, *C*, and *K* Matrices for Frequency Dependent Systems

The factor which makes the frequency domain analysis of structures most difficult is the complicated frequency dependence of the structural impedance matrix. The frequency dependence obscures the components' contributions to energy dissipation or to kinetic and potential energy storage. Knowing the role of the components helps in damping augmentation design. In this section tools will be developed for representing the complete frequency dependent impedance matrix by approximate mass, viscous damping, and stiffness matrices valid over a prescribed frequency range.

In general the impedance matrix of a structure can be represented in analytic regions as an infinite Laurent series about the origin in powers of the complex Laplace parameter s:

$$Z(s) = \sum_{n=-\infty}^{\infty} A_n s^n = A_n s^n + \cdots + A_1 s + A_0 + A_{-1}\frac{1}{s} + \cdots + A_{-n}\frac{1}{s^n} \tag{11}$$

Drawing on results from the theory of analytic functions by taking the line integral about a closed contour in the complex plane and applying Cauchy's integral theorem, the various A matrices can be evaluated exactly for a given range of s.[13] The exact calculation of the A matrices is cumbersome for complicated systems. A more useful technique is to define a truncated Laurent series:

$$\bar{Z}(s) = A_1 s + A_0 + A_{-1}\frac{1}{s} = \bar{M}s + \bar{C} + \frac{\bar{K}}{s} \tag{12}$$

and determine the constant real symmetric matrices $\bar{M}$, $\bar{C}$, and $\bar{K}$, which minimize a norm of the error $E(s)$ between the exact and approximate impedance matrices:

$$E(s) = Z(s) - \bar{Z}(s) \tag{13}$$

A cost and norm which reflect the weighted magnitude of the error matrix integrated over a prescribed path P in the complex plane can be defined:

$$J = \oint_{s=P} \|E(s)\|_E |W(s)\cdot ds| = \oint_{s=P} \mathrm{tr}\{E^H E\}|W(s)\cdot ds| \tag{14}$$

where $W(s)$ is a complex valued weighting function, and the subscript $(\ )_E$ in Eq. (14) denotes the Euclidian norm of a matrix.[14] The symbol tr{ } denotes the trace function, which is the sum of the diagonal elements of the enclosed matrix. The cost, as defined in Eq. (14), is non-negative real and can be minimized by appropriate choice of $\bar{M}$, $\bar{C}$, and $\bar{K}$ matrices.

At this point, the problem will be specialized to determining the cost along the imaginary axis. This implies that the steady-state sinusoidal error will be minimized. Assuming $s = i\omega$, the cost can be written as

$$J = \int_{\omega_1}^{\omega_2} [\mathrm{tr}\{E^H(i\omega)E(i\omega)\}\cdot|W(i\omega)|]\, d\omega \tag{15}$$

The frequency range can be chosen to minimize the cost in the vicinity of a single mode of the system or over several modes simultaneously. If several modes are within the frequency range of the integral, then the weighting function can be chosen to establish their relative importance. To determine the values of the approximate matrices which minimize the cost function, it is necessary to introduce the gradient matrix:

Suppose $f(X)$ is a scalar valued function of the elements x_{ij} of a matrix X. Then the ijth element of the gradient matrix is given by

$$\left[\frac{\partial f(X)}{\partial X}\right]_{ij} = \frac{\partial f(X)}{\partial x_{ij}} \tag{16}$$

The necessary conditions for minimum cost are that the gradient matrices of the cost with respect to the parameter matrices be equal to zero.[15] For the cost defined by substituting Eqs. (12) and (13) into Eq. (15), the three necessary conditions are that the gradients with respect to the parameter matrices are zero as indicated in Eq. (17) below. The gradient matrices can be calculated by employing the formulas given in Appendix A of Ref. 15. They yield three coupled integral matrix equations for the approximate mass, stiffness, and damping matrices:

$$\frac{\partial J}{\partial \bar{M}} = 0 = \mathrm{Re}\left\{\int_{\omega_1}^{\omega_2} [i\omega Z(i\omega) + \bar{M}\omega^2 - i\omega\bar{C} - \bar{K}]|W(i\omega)|\, d\omega\right\}$$

$$\frac{\partial J}{\partial \bar{C}} = 0 = \mathrm{Re}\left\{\int_{\omega_1}^{\omega_2} \left[-Z(i\omega) + \bar{M}i\omega + \bar{C} - \frac{\bar{K}}{i\omega}\right]|W(i\omega)|\, d\omega\right\}$$

$$\frac{\partial J}{\partial \bar{K}} = 0 = \mathrm{Re}\left\{\int_{\omega_1}^{\omega_2} \left[\frac{Z(s)}{i\omega} - \bar{M} - \frac{\bar{C}}{i\omega} + \frac{\bar{K}}{\omega^2}\right]|W(i\omega)|\, d\omega\right\} \tag{17}$$

These matrix equations can be solved for the approximate $\bar{M}$, $\bar{C}$, and $\bar{K}$:

$$\bar{M} = \frac{1}{ab - c^2}\left[b\int_{\omega_1}^{\omega_2} \omega Z_I(i\omega)|W(i\omega)|\, d\omega - c\int_{\omega_1}^{\omega_2} \frac{1}{\omega} Z_I(i\omega)|W(i\omega)|\, d\omega\right]$$

$$\bar{C} = \frac{1}{c}\int_{\omega_1}^{\omega_2} Z_R(i\omega)|W(i\omega)|\, d\omega$$

$$\bar{K} = \frac{1}{ab - c^2}\left[c\int_{\omega_1}^{\omega_2} \omega Z_I(i\omega)|W(i\omega)|\, d\omega - a\int_{\omega_1}^{\omega_2} \frac{1}{\omega} Z_I(i\omega)|W(i\omega)|\, d\omega\right] \tag{18}$$

where

$$a = \int_{\omega_1}^{\omega_2} \omega^2 |W(i\omega)|\, d\omega$$

$$b = \int_{\omega_1}^{\omega_2} \frac{1}{\omega^2}|W(i\omega)|\, d\omega$$

$$c = \int_{\omega_1}^{\omega_2} |W(i\omega)|\, d\omega$$

$$Z(i\omega) = Z_R(i\omega) + iZ_I(i\omega) \tag{19}$$

Equations (18) and (19) provide a useful tool for identifying approximate mass, stiffness, and damping matrices from a system's global impedance matrix without foreknowledge of the functional frequency dependence of the impedance; only the values of the impedance are needed.

Several observations on the approximate system matrices can be drawn from Eq. (18). First, the equations indicate that any elements of the global impedance matrix which have the frequency dependence of the terms in Eq. (5) will be represented exactly in the approximate $\bar{M}$, $\bar{C}$ and $\bar{K}$ matrices. The equations are only free to "interpret" those elements which have more complicated frequency dependence, such as resonant components, which have internally condensed dynamics or elements with inherently more complex frequency dependence such as viscoelastics.

Second, it is important to note that the dimensions of the approximate matrices are the same as the dimensions of the impedance matrix. The dimensions of the impedance matrix are, however, typically less than the order of the system being modeled because of reduced component dynamics or the frequency dependence of elements. Therefore, the approximate system will in general have a lower order than the actual system. This has implications concerning the ability of the approximate system to accurately represent the actual system.

It limits the number of modes of the actual system which can be accurately represented by the approximate system. In principle, the approximate system can represent the same number of modes as the dimension of the approximate ma-

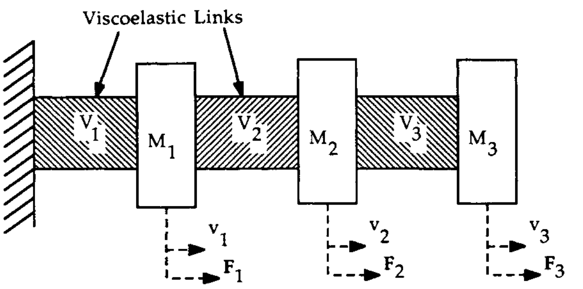

Fig. 1 Example 3-DOF mechanical system with viscoelastic links V_1, V_2, and V_3.

trices (which may be less than the number of resonant modes of the actual system). In practice, the number of resonant modes of the actual system which can be accurately modeled is smaller than the order of the approximate system. The remaining DOF available in the approximate system are devoted to helping model effects of residual dynamics. As a rule of thumb, the designer should include no more resonant modes than half the dimension of the approximate system and choose the frequency range of the fit accordingly.

The approximate damping matrix is proportional to the average of the real part of the impedance matrix. Thus effects other than viscous can contribute to $\bar{C}$. For example, the impedance of a resonant subsystem, such as a proof mass damper, can have a contribution to the real part of the impedance matrix and thus to the approximate damping matrix. It should also be noted that as the frequency range over which the damping matrix integral is evaluated is decreased, the approximate damping matrix becomes identically equal to the real part of the global impedance matrix evaluated at a single frequency. The approximate mass and stiffness matrices depend on the frequency weighted integrals of the imaginary part of the impedance matrix.

Equation (18) can be used to identify components before their assembly into the global impedance matrix or applied to the global impedance matrix directly. In the following section, these techniques for approximate model building will be applied to a representative structural system to examine their ability to accurately represent a frequency dependent system.

Example of Approximate Model Building

System Representation and Condensation

The dynamics of the 3-DOF system presented in Fig. 1 will be modeled to illustrate the approximate procedures for model generation presented in the previous section. This system has many of the characteristics representative of typical structural frequency domain models. First, the system contains viscoelastic elements with relatively complicated frequency dependence. Second, although there are 3 DOF in the full system model, in this example the system will be represented by the forces and velocities at only the first and third nodes. The second node will be reduced out of the full model to better represent a system with internal component dynamics.

To start, the impedances of the inertial and viscoelastic components can be assembled into a 3 × 3 global impedance matrix for nodes 1, 2, and 3.

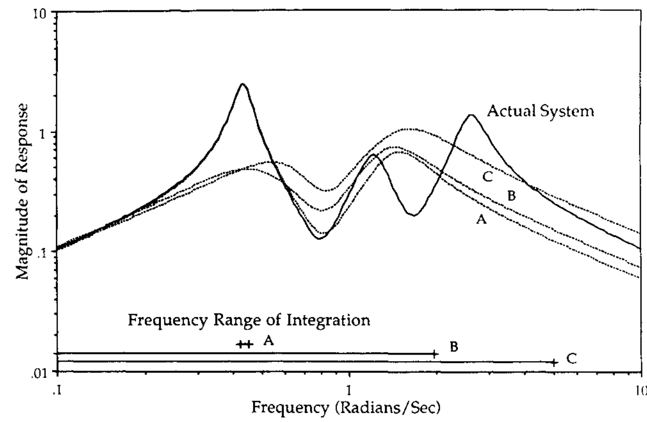

Fig. 2 Convergence of the approximate transfer function as the range of frequency over which it is calculated is narrowed about the first mode.

$$\begin{bmatrix} M_1 s + Z_{V1}(s) + Z_{V2}(s) & -Z_{V2}(s) & 0 \\ -Z_{V2}(s) & M_2 s + Z_{V2}(s) + Z_{V3}(s) & -Z_{V3}(s) \\ 0 & -Z_{V3}(s) & M_3 s + Z_{V3}(s) \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \tag{20}$$

where the values for the mass and viscoelastic parameters are chosen as

$$M_1 = M_3 = 1, \qquad M_2 = 2 \tag{21}$$

$$Z_{V1}(s) = Z_{V3}(s) = \frac{1 + 0.4s}{s + 0.2s^2}, \qquad Z_{V2}(s) = \frac{2 + 4s}{s + s^2} \tag{22}$$

At this point we will assume that there is no forcing or velocity information available at node 2. To derive the model for such a system, this DOF can be eliminated using matrix condensation [presented in Eq. (8)] to obtain a 2 × 2 global impedance matrix for the system dynamics as observed at nodes 1 and 3:

$$\begin{bmatrix} M_1 s + Z_{V1}(s) + \dfrac{Z_{V2}(s)[M_2 s + Z_{V3}(s)]}{M_2 s + Z_{V3}(s) + Z_{V2}(s)} & -\dfrac{Z_{V2}(s) Z_{V3}(s)}{M_2 s + Z_{V3}(s) + Z_{V2}(s)} \\ -\dfrac{Z_{V2}(s) Z_{V3}(s)}{M_2 s + Z_{V3}(s) + Z_{V2}(s)} & M_3 s + \dfrac{Z_{V3}(s)(M_2 s + Z_{V2}(s))}{M_2 s + Z_{V3}(s) + Z_{V2}(s)} \end{bmatrix} \begin{bmatrix} v_1 \\ v_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_3 \end{bmatrix} \tag{23}$$

This is the impedance matrix of the actual system which will be approximated. Notice that the above global impedance matrix is of dimension 2 while representing a system of order 9 (3 resonant modes and 3 first-order viscoelastics).

Approximate $\bar{M}$, $\bar{C}$, and $\bar{K}$ matrices will now be developed for the above impedance matrix, first in the vicinity of the first mode and then in the vicinity of the third mode using Eq. (18). In calculating the integrals in Eq. (18), a weighting of $1/\omega$ was used to give uniform weighting on a log scale. The integrals for the approximate matrices were evaluated numerically.

Figure 2 shows the response at node 1 due to unity forcing at node 1 of the actual system and several approximate systems. The $\bar{M}$, $\bar{C}$, and $\bar{K}$ matrices for the approximate system were determined by evaluating the integrals of Eq. (18) for various frequency ranges centered about the first mode and over a progressively smaller range. From Fig. 2, it is clear that the response of the approximate systems more closely replicates the actual response for the first mode as the frequency range is narrowed about that mode. The approximate response at the first mode improves dramatically once the second mode is outside the range of integration.

Figure 3 shows the convergence of the poles of the approximate system to the actual system poles as the frequency range is progressively decreased about the first mode just as it was in Fig. 2. As expected, the approximate system has only two resonant poles, the lower of which converges rapidly to the actual first mode poles as the frequency range narrows around the first mode. When the frequency range includes the second mode, the method attempts with limited success to represent that mode as well as the first mode.

The approximate matrices are effectively converged to the first mode when the frequency range for the integrals of Eq. (18) is from 0.38 to 0.51 rad/s

$$\bar{Z}(s) = \begin{bmatrix} 2.12 & 0.65 \\ 0.65 & 1.00 \end{bmatrix} s + \begin{bmatrix} 0.40 & -0.23 \\ -0.23 & 0.28 \end{bmatrix} + \begin{bmatrix} 1.71 & -0.67 \\ -0.67 & 0.67 \end{bmatrix} \frac{1}{s} \tag{24}$$

This approximate model will be used in later sections to examine approximate methods of determining the first mode natural frequencies and damping. It should be noted that the application of Eq. (18) yields symmetric system matrices and a distributed mass matrix.

Figures 4 and 5 show the system response and pole locations for approximate systems with the frequency range of integration converging on the third mode. The third mode was chosen as a challenging case because it is in the range of rapidly changing viscoelastic properties and is highly coupled to the second mode. These figures show that the approximate model response closely approaches the actual third mode response, but the pole location does not achieve complete convergence as the frequency range is narrowed. This is due to the relatively high damping and interaction with the second mode.

These limitations point to some guidelines for choosing the frequency range of integration. First, for accurate modal models of systems with component dynamics, the number of modes within the frequency range of integration should be about half the dimension of the approximate matrices. As an example, in order to converge completely to the first mode pole, it was necessary to include only that pole in the frequency range of integration. This is in accord with the rule of thumb since the approximate system dimension is 2. Second, for good convergence to a specific mode, that mode must contribute significantly to the system impedance within the frequency range of integration.

Approximate Analysis of Linear Damped Systems

Motivation for Approximate Expressions for Dynamic Quantities

The previous sections concentrated on modeling structures with impedances and developing approximate $\bar{M}$, $\bar{C}$, and $\bar{K}$ models for the system. Once the approximate system model has been obtained, the poles of this approximate model can be found exactly by performing a generalized eigenanalysis to obtained damped system poles and possibly complex mode shapes. Since the intent of this paper is to develop techniques for preliminary analysis, the following sections will concentrate on presenting a simple approximate method for calculating the natural frequencies and damping which can be applied to any system in the $\bar{M}$, $\bar{C}$, and $\bar{K}$ matrix form. This approximate method is presented as an alternative to the exact eigenanalysis for cases where the exact analysis may be inappropriate, such as preliminary design, or where a simple method can convey more physical insight into the damping phenomena.

In the following sections, the problem of analyzing $\bar{M}$, $\bar{C}$, and $\bar{K}$ models for approximate natural frequencies and damping ratios will be examined. The first step in the analysis is to turn back to the frequency domain impedance model to gain insight into energy dissipation and storage mechanisms. It is difficult to determine the exact system natural frequencies and damping from the actual impedance model because of the frequency dependence of the global impedance matrix. In general a natural mode of the linear system occurs when the

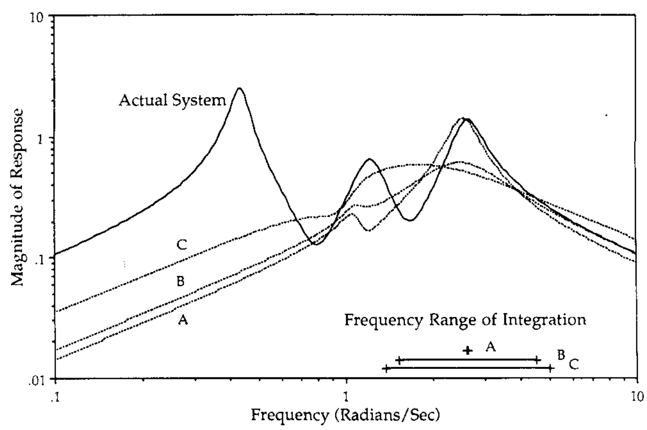

Fig. 4 Convergence of the approximate transfer function as the range of frequency over which it is calculated is narrowed about the third mode.

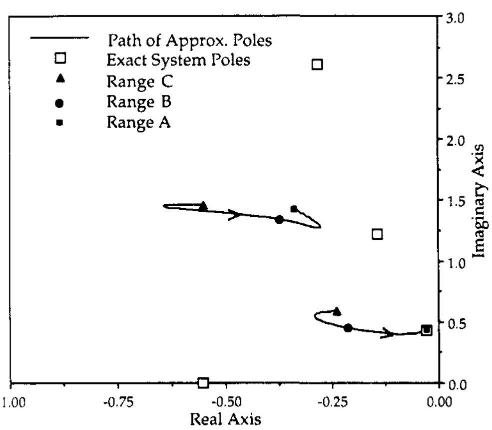

Fig. 3 S-plane plot of the poles of the actual and approximate systems as the range of frequency over which the approximate system is calculated is narrowed about the first mode; systems (A), (B), and (C) correspond to those of Fig. 3.

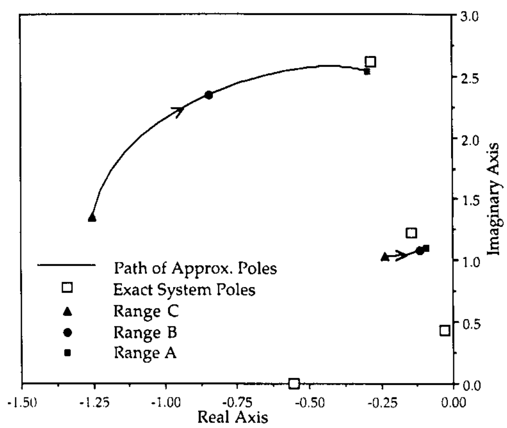

Fig. 5 S-plane plot of the poles of the actual and approximate systems as the range of frequency over which the approximate system is calculated is narrowed about the third mode; systems (A), (B), and (C) correspond to those of Fig. 4.

determinant of the impedance matrix equals zero:

$$|Z_L(s)| = 0 \tag{25}$$

The process of finding the solutions in the complex plane is complicated by several factors and usually requires a numerical procedure such as the Wittrick-Williams algorithm[16] for undamped frequency-dependent systems or Muller's method for the determination of complex zeros.[17] One complicating factor is that $Z(s)$ is often a transcendental, nonlinear function of the complex frequency s. Another complicating factor is that for higher-order systems, the determinant of the impedance matrix can become extremely large causing computational difficulties. These difficulties can be overcome by searching for minima of the natural log of the magnitude of the determinant instead of the determinant itself. This makes the problem more amenable to traditional numerical search routines.

This section presents a method for determining approximate system natural frequencies and damping, which are useful as stand-alone approximations or as initial values in the numerical search for the exact solutions. The first step in determining the approximate modes is to premultiply Eq. (4) by v^H giving a complex scalar equation:

$$v^H Z v = v^H F_{\text{EXT}} \tag{26}$$

This equation is analogous to a frequency domain form of Tellegen's theorem for electrical networks.[18] Tellegen's theorem can be applied to mechanical systems by virtue of the analogy between Kirchhoff's current and voltage laws and force equilibrium and displacement compatibility, respectively. The theorem is an expression of energy conservation and relates the power input from external forces with the power dissipated in the system components. The frequency domain representation of power can be divided into a real part, termed the "real power," and an imaginary part, termed the "reactive power." Tellegen's theorem in the frequency domain is sometimes referred to as the "conservation of real and reactive power."

The physical significance of Eq. (26) can be found by examining its real and imaginary terms. Multiplying Eq. (26) by a half and dividing the products into their real and imaginary parts, yields

$$P_L + iX_L = P_{\text{EXT}} + iX_{\text{EXT}} \tag{27}$$

where P_L and X_L are the real and imaginary parts of the quadratic product of the structural impedance matrix and the velocity mode shape, which can be expanded:

$$P_L + iX_L = \frac{1}{2} v^H Z v = \frac{1}{2}(v_R - iv_I)^T(Z_R + iZ_I)(v_R + iv_I) \tag{28}$$

Assuming a symmetric impedance matrix P_L and X_L reduce to

$$P_L = \frac{1}{2} v^H Z_R v \tag{29}$$

$$X_L = \frac{1}{2} v^H Z_I v \tag{30}$$

Notice that P_L and X_L involve only the real and imaginary parts of the global impedance matrix, respectively. The assumption of symmetry of Z will not limit the derivation's applicability to damped structural systems since systems with passive structural damping and component resonances maintain reciprocity.

Equation (29) leads to some interesting conclusions concerning damping in structural systems. By Eq. (18), the system's approximate damping matrix in a particular frequency range is an average of the real part of the system impedance matrix. It can be concluded from Eq. (29) that the real power dissipated in the system is dependent on two factors: the size of the real part of the system impedance matrix and its placement in relation to the system mode shape at that frequency. Contributions to the real part of the impedance matrix and therefore the dissipation of real power are made by both the imaginary part of a complex modulus, $E(1 + i\eta)$, the real impedance of any resonant subsystem, and any viscous damping terms in the impedance. On the other hand, the system mass and stiffness matrices defined by Eq. (18) are related to the imaginary part of the system impedance and thus to the reactive power.

Approximate Formulas for System Frequency and Loss Factors

In design and preliminary analysis, it is desirable to have simple ways of determining approximate natural frequencies and damping. In this section, approximate natural frequencies and damping for a system represented in the frequency domain will be derived.

The first step in determining the approximate natural frequencies and damping is to represent the system with the approximate matrices as determined in a previous section. Within a narrow frequency band, the impedance matrix of a system can be represented by equivalent mass, stiffness, and damping matrices as given by Eq. (18). Representing a system by the form of Eq. (12), substituting this into Eq. (26), and assuming no external forcing, gives

$$v^H\left(\bar{M}s + \bar{C} + \frac{\bar{K}}{s}\right)v = 0 \tag{31}$$

If it is assumed that the solution for s has the form

$$s = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \tag{32}$$

then substituting Eq. (32) into Eq. (31) and solving for the natural frequency and damping ratio of the system modes from the real and imaginary parts of Eq. (31) yields

$$\omega_n^2 \cong \frac{v^H \bar{K} v}{v^H \bar{M} v} \tag{33}$$

$$\zeta \cong \frac{v^H \bar{C} v}{2 v^H \dfrac{\bar{K}}{\omega_n} v} \tag{34}$$

The equations are similar to Rayleigh quotients for the natural frequency of an undamped system,[19] and the equations become exact if the exact complex mode shape is used and if the complex impedance matrix can be exactly represented by the $\bar{M}$, $\bar{C}$, and $\bar{K}$ matrices at the mode of interest.

The energy dissipation of a system can be more generally defined at frequencies other than resonance by using the loss factor. The loss factor is defined as the ratio of energy dissipated in a cycle E_{dis} divided by the peak system strain energy in the cycle U and can be expressed in terms of the damping coefficient as

$$\eta \equiv \frac{E_{\text{dis}}}{2\pi U} = 2\zeta\frac{\omega}{\omega_n} \tag{35}$$

Substituting Eq. (34) into Eq. (35) gives a simple approximate expression for the system loss factor

$$\eta \cong \frac{v^H \bar{C} v}{v^H \dfrac{\bar{K}}{\omega} v} \tag{36}$$

where ω is the frequency at which the loss factor is desired.

Equations (33), (34), and (36) allow the designer to make a quick estimate of the system natural frequencies and damping for an assumed or experimentally determined mode shape. The mode shape can be pure real for simplicity or complex if a more accurate value of the damping is desired. The issue of what mode shape to use will be examined in the following sections. Since any number of damping mechanisms can contribute to the approximate system damping matrix $\bar{C}$, the approximate expressions for damping ratio and loss factor are useful in evaluating diverse damping schemes.

System Sensitivities to Parameter Changes

Because the method of determining the approximate frequencies and damping depends on the approximate system matrices and assumed mode shapes, it is important to understand the sensitivities of the frequency and damping to variations in these quantities. The mode-shape variations will be examined first.

To determine the sensitivities of the system frequencies and damping ratios to variations in the assumed mode shapes in Eqs. (33), (34), and (36), the variation for any ratio of quadratics will first be determined. Assuming a general form

$$y = \frac{x^H A x}{x^H B x} \tag{37}$$

and letting

$$y = y_0 + \delta y, \qquad x = x_0 + \delta x \tag{38}$$

a solution for δy neglecting second-order terms can be obtained for mode-shape variations:

$$\frac{\delta y}{y_0} \cong 2\,\mathrm{Re}\left\{\frac{x_0^H (A - y_0 B)\delta x}{x_0^H A x_0}\right\} = 2\,\mathrm{Re}\left\{\frac{x_0^H A \delta x}{x_0^H A x_0} - \frac{x_0^H B \delta x}{x_0^H B x_0}\right\} \tag{39}$$

Applying this general formula to the approximate expressions of Eqs. (33) and (34) leads to the sensitivity of the frequency to variations in the assumed mode shape [from Eq. (33)]

$$\frac{\delta \omega_n^2}{\omega_n^2} = 2\,\mathrm{Re}\left\{\frac{v^H(\bar{K} - \omega_n^2 \bar{M})\delta v}{v^H \bar{K} v}\right\} \tag{40}$$

and to the sensitivity of the damping ratio [from Eq. (34)]

$$\frac{\delta \zeta}{\zeta} = 2\,\mathrm{Re}\left\{\frac{\boldsymbol{v}^H\left(\bar{\boldsymbol{C}} - \zeta \dfrac{\bar{\boldsymbol{K}}}{\omega_n}\right)\delta \boldsymbol{v}}{\boldsymbol{v}^H \bar{\boldsymbol{C}} \boldsymbol{v}}\right\} \tag{41}$$

The sensitivities for the loss factor, Eq. (36), can be found in a similar manner.

Since the numerator of the quotients in Eqs. (40) and (41) do not equal zero for variations about the exact mode shape $\boldsymbol{v}^H$, the approximate natural frequency and damping, Eqs. (33) and (34), do not necessarily possess the insensitivity properties of the Rayleigh quotient[19] to first-order variations in the mode shape. They can, however, be relatively insensitive to variations in the imaginary part of the mode shape. The sensitivities of the natural frequency and damping to variations in the imaginary part of the mode shape depend only on the product of the imaginary part of the variation with that of the exact mode shape. Since the imaginary part of the exact mode shape is typically small for lightly damped systems, the natural frequency and damping are relatively insensitive to variations in this part of the mode shape.

In the limiting case of no system damping, the numerator of Eq. (40) for the natural frequency sensitivity is zero. In this case the natural frequency for Eq. (33) becomes identical to the Rayleigh quotient and shares with it the property of being insensitive to first-order variations in the real part of the mode shape. The expression for damping variation, Eq. (41), is very sensitive to changes in the real part of the assumed mode shape. This gives some freedom in the choice of the assumed mode shape. In particular, it implies that the real part of a slightly complex mode shape can be used in Eqs. (33) and (34) to obtain a reasonable estimate of modal damping.

The sensitivities of the system frequencies and loss factors to variations in the approximate system matrices will now be examined. Assuming the general form of Eq. (37) and allowing variations

$$y = y_0 + \delta y, \qquad \boldsymbol{A} = \boldsymbol{A}_0 + \delta \boldsymbol{A}, \qquad \text{and } \boldsymbol{B} = \boldsymbol{B}_0 + \delta \boldsymbol{B} \tag{42}$$

a solution for δy neglecting second-order terms can be obtained

$$\frac{\delta y}{y_0} \cong \frac{\boldsymbol{x}^H \delta \boldsymbol{A} \boldsymbol{x}}{\boldsymbol{x}^H \boldsymbol{A}_0 \boldsymbol{x}} - \frac{\boldsymbol{x}^H \delta \boldsymbol{B} \boldsymbol{x}}{\boldsymbol{x}^H \boldsymbol{B}_0 \boldsymbol{x}} \tag{43}$$

Applying this general formula to the approximate quadratic expressions for system loss factor and frequency leads to the sensitivity of the natural frequency to variations in the approximate system matrices [from Eq. (33)]

$$\frac{\delta \omega_n^2}{\omega_n^2} = \frac{\boldsymbol{v}^H \delta \bar{\boldsymbol{K}} \boldsymbol{v}}{\boldsymbol{v}^H \bar{\boldsymbol{K}} \boldsymbol{v}} - \frac{\boldsymbol{v}^H \delta \bar{\boldsymbol{M}} \boldsymbol{v}}{\boldsymbol{v}^H \bar{\boldsymbol{M}} \boldsymbol{v}} \tag{44}$$

and of the damping ratio [from Eq. (34)]

$$\frac{\delta \zeta}{\zeta} = \frac{\boldsymbol{v}^H \delta \bar{\boldsymbol{C}} \boldsymbol{v}}{\boldsymbol{v}^H \bar{\boldsymbol{C}} \boldsymbol{v}} - \frac{\boldsymbol{v}^H \delta \bar{\boldsymbol{K}} \boldsymbol{v}}{\boldsymbol{v}^H \bar{\boldsymbol{K}} \boldsymbol{v}} \tag{45}$$

Equations (44) and (45) can be shown to be equivalent to results obtained through traditional eigenvalue perturbation schemes.[20]

The natural frequency and damping ratio are sensitive to variations in the system matrices. The sensitivity depends on how the variation is weighted by the mode shape, e.g., in areas of high strain, changes in stiffness can produce large changes in frequency. Equation (45) can be useful for determining the best placement of damping treatments and devices.

Example of Approximate Analysis of a Damped System

Sensitivities of Exact and Approximate Frequencies and Damping Ratios

In this section the sensitivities of the approximate formulas for frequency and damping ratio to choice of approximate mode shapes will be examined. Once the approximate $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{C}}$, and $\bar{\boldsymbol{K}}$ matrices which best represent the system in the vicinity of a mode have been determined, Eqs. (33) and (34) can be applied to determine the approximate modal frequency and damping ratio. For accurate results, Eq. (33) requires knowledge of the system mode shape a priori. Since the system mode shape is unknown, a "best guess" must be made from either the forced system response velocities in the vicinity of the resonance or from an assumed mode shape. The sensitivity to errors in these guesses will now be investigated.

The sensitivities of the approximate natural frequency and damping ratio [Eqs. (33) and (34)] of the first mode of the sample problem to variations in the real and imaginary components of the assumed mode shape were calculated. For this calculation the mode shape was assumed to have the form of a perturbation of the exact mode shape

$$\boldsymbol{v} = \boldsymbol{v}_0 + \delta \boldsymbol{v} = \begin{bmatrix} 1 \\ 1.63 + i0.08 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.63\delta_R + i0.08\delta_I \end{bmatrix} \tag{46}$$

where δ_R represents the real deviation from the exact mode shape and δ_I represents the imaginary deviation. The approximate matrices valid for the first mode from Eq. (24) were used for this example.

Figure 6 shows the results of these calculations for the sensitivity of natural frequency and damping ratio to real variations in the mode shapes (δ_R varied, $\delta_I = 0$). Line A

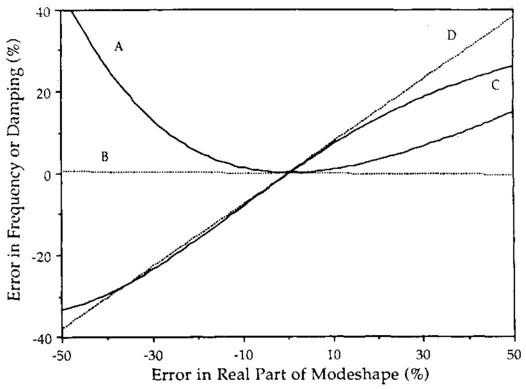

Fig. 6 Effect of variations in the real part of the assumed mode shape on the accuracy of the approximate natural frequency (A) and damping ratio (C) compared to the sensitivities predicted by Eqs. 40 (B) and 41 (D), respectively.

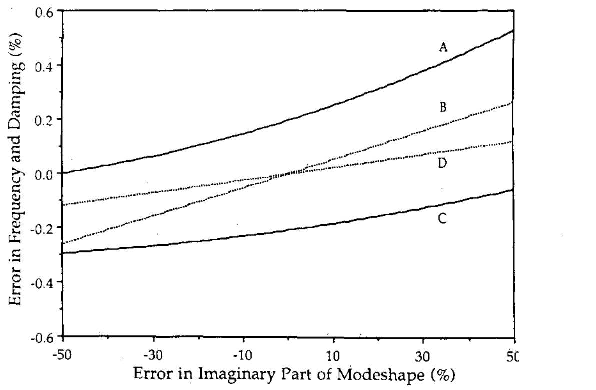

Fig. 7 Effect of variations in the imaginary part of the assumed mode shape on the accuracy of the approximate natural frequency (A) and damping ratio (C) compared to the sensitivities predicted by Eqs. 40 (B) and 41 (D), respectively.

shows the percent error in the square of the natural frequency ($\delta\omega_n^2/\omega_n^2$) as calculated by Eq. (33) as a function of δ_R. Line B shows the error in frequency predicted from Eq. (40) due to first-order variations in the real part of the mode shape. Line B has the slope of line A at zero perturbation. Line C represents the percent error in the damping ratio ($\delta\zeta/\zeta$) as calculated by Eq. (34) as a function of δ_R. Line D shows the error in damping ratio due to real variations in the mode shape as predicted from Eq. (41). Figure 7 shows the results of these calculations for the sensitivity of natural frequency and damping ratio to imaginary variations in the mode shapes ($\delta_R = 0$, δ_I varied). The various lines represent the same quantities as for Fig. 6 but due to variations in the imaginary part of the mode shapes.

Figure 6 shows that the natural frequency is relatively insensitive to variation in both the real and imaginary components of the assumed mode shape. It should be noted that, in general, for damped systems, the exact mode shape does not correspond to a local minimum of the natural frequency as it does for the Rayleigh quotient of an undamped system. Figure 6 also shows that the damping ratio is very sensitive to variations in the real mode shape. Figure 7 shows that for this system both the approximate frequencies and damping are insensitive to changes in the imaginary part of the mode shape. This leads to the conclusion that a pure real mode shape can be used to obtain accurate estimates of both the natural frequencies and damping of damped structures using Eq. (33). The real part of the mode shape can be taken from an undamped system eigenanalysis. Figures 6 and 7 also indicate that Eqs. (40) and (41) can be effectively used to find the sensitivities of these quantities to the choice of mode shape.

Implementation Issues

In this section, some aspects of the implementation of the frequency domain structural modeling will be discussed. The general method used to model structures with frequency dependent impedances as well as techniques for finding approximate frequencies and damping will be reviewed.

System Modeling

The method for modeling structures using the frequency dependent impedances of the elements resembles that for assembling and solving a conventional finite-element model. It proceeds as follows.

1) The impedance matrices of the individual components are evaluated at a desired frequency.

2) The matrices are then rotated into the global coordinate frame.

3) The matrices are assembled into a global impedance matrix as are the element stiffness matrices in conventional finite-element models.

4) The restrained DOF are then eliminated, and the global impedance matrix is reduced to contain only the retained set of DOF.

5) For a given forcing, the retained velocities can then be determined by inverting the reduced impedance matrix.

This process is essentially identical to solutions for the static displacement using a finite element model. The element stiffnesses are replaced with the element impedances evaluated at a single frequency. In this way, the system response for the retained DOF set is obtained. The system response as a function of frequency is created by performing the solution procedure repetitively for multiple frequencies.

Algorithms for Approximate System Dynamic Quantities

There are several difficulties associated with the implementation of the approximate formulas for frequencies and damping presented in Eqs. (33) and (34). These problems stem from the fact that the matrices used in the approximate formulas are dependent on the frequencies at which they are evaluated. The algorithm for determining the approximate matrices follows.

1) The initial estimate for the natural frequency of the system is found by inspection of the transfer function derived in the manner described above. The transfer function can be computed over a broad frequency range and inspected for local maxima.

2) The desired maxima are bracketed, and the approximate mass, damping, and stiffness matrices for the system are calculated using Eq. (18).

3) These stiffness and mass matrices are used along with an assumed approximate mode shape (possibly the forced response at the desired frequency) in Eq. (33) to estimate the modal natural frequency.

4) This estimate of the frequency is used in calculation of the damping from Eq. (34) or (36).

From these approximate estimates of the modal damping and natural frequency, the approximate pole locations can be computed and used as an initial guess for the exact pole search.

Conclusions

In this paper, techniques for an approximate frequency domain analysis of damped mechanical systems have been presented. Mechanical impedance was used to model the structural system since it could incorporate frequency dependent components, such as viscoelastic materials or resonant substructures, more simply than equivalent time domain representations. Component impedances could be represented by either analytical models or experimental data.

A method was presented for determining an optimal mass, stiffness, and damping matrix representation for a system described by a complicated, frequency dependent impedance matrix. The expressions for approximate $\bar{M}$, $\bar{C}$, and $\bar{K}$ matrices took the form of weighted frequency integrals of the impedance matrix. The damping matrix was found to depend solely on the real part of the impedance matrix; while the imaginary part influenced the approximate mass and stiffness matrices. The approximate matrices could represent more than one system mode if an appropriate frequency weighting function were chosen.

Approximate formulas were derived for system natural frequency and damping, which took the form of Rayleigh's quotient in the limiting case of no system damping. The sensitivities of the natural frequency and damping ratio (or loss factor) to variations in the assumed mode shape or system matrices were derived and found to be insensitive to variations of the imaginary part of the assumed mode shape. In addition, the approximate natural frequency was insensitive to real part variations; while the approximate damping was not. Real modes can therefore be used in initial damping estimation instead of complex modes. The variations of the approximate natural frequency and damping due to variations of the system matrices were found. These expressions can be used in place of time domain eigenvalue perturbation analyses.

The algorithms for implementation of the approximate frequency domain modeling and analysis were similar to conventional time domain finite element modeling and static solution but required the generation of a system impedance matrix at many frequencies. Implementation of the algorithms on a sample problem illustrated several points. First, the approximate model was capable of representing the actual system well over a broad frequency range. Second, the sensitivities of the approximate natural frequency and damping to mode shape variations agreed well with predictions. The formulas for sensitivities to variations in the assumed mode shapes or system matrices are useful tools for damped system design. Coupled with the method for determining the system approximate matrices, they form the basis for a unified design methodology for damped structures.

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